Revisiting Goldstone's theorem

Recollections

After so many years, we have tried repeatedly, but failed to write a paper together!

Met 1st time (Madrid 1981?)

Boston, DZF, BU, BGR, CFT, String Theory…

Return to Europe

Benasque, N=2 SW…

Neural nets, computational complexity, quantum computing

1905 Centennial, public talks

Failed to go to El Bulli…

MEP (Entanglement) back to the USA

“Never underestimate the pleasure people get when they listen to something they already know”. (E. Fermi)

Motivation

A few years ago, Hellerman, Orlando, Reffert and Watanabe studied in detail the behavior of the Wilson-Fisher fixed point for O(2) in the limit of large charge. In other words, they computed some of the anomalous dimensions and properties around the fixed point by expanding in large charges.

Our work together tries to understand why such approach, which one would think is doomed to failure, works much better than expected, as shown by numerical simulations. Work together with Orlando, Reffert and Loukas (and thanks to Orestis for help with the slides).

This led us to the study of novel ways of understanding symmetry breaking patterns in QFT, the appearance of NR behavior at low energies, and in some limits (large charge) a possible systematic way of computing some properties of CFT's

Many of our arguments are heuristic, but we are encouraged by the reasonable agreement with MC computations

RG Flows

CFT play a central role in QFT

Fixed points of RG flows

Critical phenomena

String theory & Q-gravity

Long distance: IR fixed points

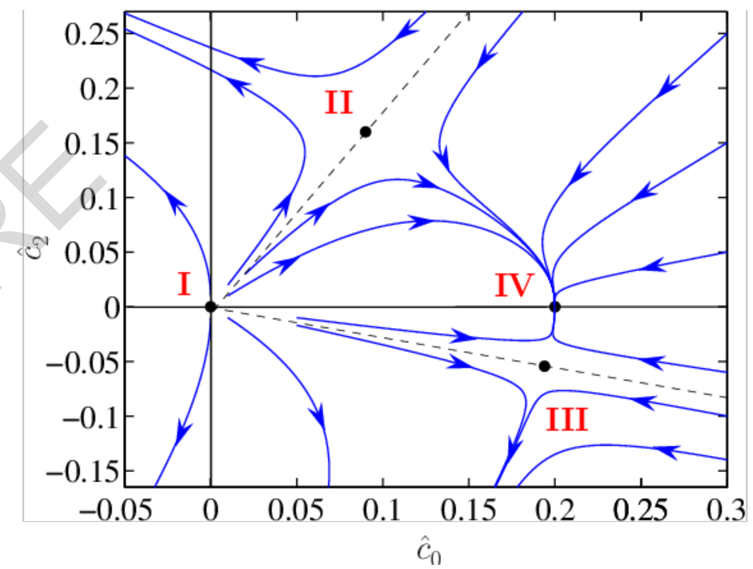
Short distance: UV fixed points

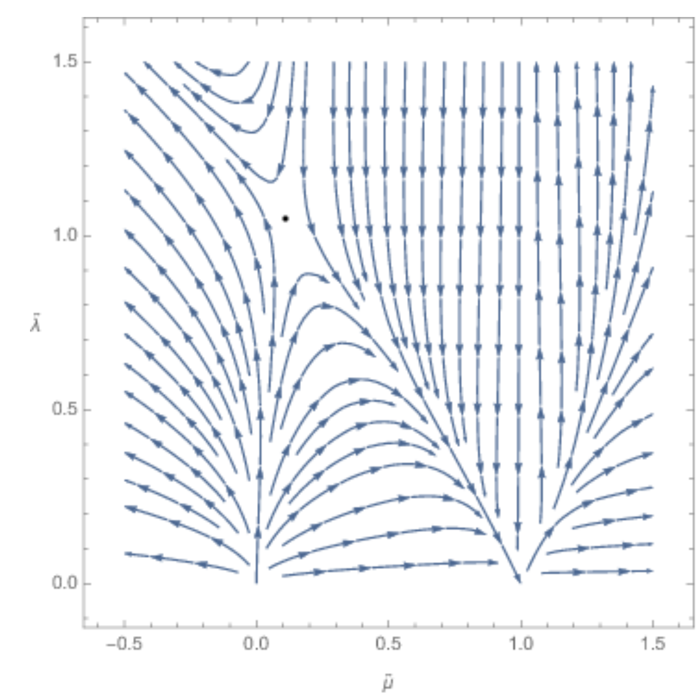
Description near fixed points well defined

In-between: messy, unclear except d=2

Zamolodchikov's metric, monotonicity

Why only fixed points…





RG Flows

The problem is that most CFTs do not have small parameters generically

Lattice computations, Non-perturbative methods, whenever possible.

Bootstrap in its new incarnation is quite successful

Are there other methods that could qualitatively help us understand generic features of IR CFTs?

We explore the large charge sectors of theories with global symmetries.

This is similar to the case of Rydberg states in atoms, or the limit of large quantum numbers (Sommerfeld, Maslov, WKB…)

Symmetries

In relativistic, local QFT, we are familiar with the WW or NG realization of global symmetries. The general properties are well understood.

In the context of NR theories, symmetry breaking patterns are rather rich, and we may still be far from a complete understanding.

We have been looking at a kind of intermediate case, which is the study of states of a QFT/CFT with large values of the global charge. Many of our results are heuristic, but they are confirmed by "exact" lattice computations.

Goldstone I

In the relativistic case, a local unitary QFT satisfies Coleman's theorem. This shows the strong consequences of these assumptions, in particular CPT and the inevitable existence of antiparticles:

Coleman's theorem:

The symmetries of the vacuum are the symmetries of the world.

Given a current, not necessarily conserved, assume the associated charge annihilates the ground state,

$$Q = \int j^0(t,x)\,dx, \qquad Q|0\rangle = 0$$

$$\Downarrow$$

(Using Federbush-Johnson)

$$\partial_\mu j^\mu = 0$$

This is not true in the NR case, particles and antiparticles split

Basic relativistic example

Work in a space box of side L and PBC

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi \qquad \phi \to \phi + \mathrm{const} \qquad j_\mu = \partial_\mu\phi$$

$$[iQ, \phi(x)] = 1 \qquad e^{i\xi Q}\phi(x)e^{-i\xi Q} = \phi(x) + \xi$$

$$\phi(x,t) = \phi_0 + t\,\pi_0 + \sum_{k\neq 0}\frac{1}{\sqrt{2Vk}}\left(a(k)e^{-ik.x} + a(k)^+e^{ik.x}\right)$$

$$a = \frac{1}{\sqrt{2}}(\phi_0 + iL\pi_0), \quad a^+ = \frac{1}{\sqrt{2}}(\phi_0 - iL\pi_0) \quad [a, a^+] = \frac{1}{L^2}, \quad Q = \frac{L^2}{i\sqrt{2}}(a - a^+)$$

$$a|0\rangle = 0, \quad a(k)|0\rangle = 0, \quad \langle 0|\phi(x,t)|0\rangle = 0, \quad |\xi\rangle = e^{-i\xi Q}|0\rangle$$

$$\langle 0|\xi\rangle = e^{-\frac{1}{4}\xi^2 V^{\frac{D-2}{D-1}}}\langle 0|0\rangle$$

Basic non-relativistic example

Work in a space box of side L and PBC, and consider the Schroedinger field theory, whose symmetry group is ISO(2):

$$\mathcal{L} = i\psi^+ \frac{\partial}{\partial t}\psi - \frac{1}{2m}\nabla\psi^+ \cdot \nabla\psi$$

$$\psi(x,t) = \frac{1}{\sqrt{V}}\sum_k a(k) e^{-i(\epsilon(k)t - k\cdot x)}$$

$$\epsilon(k) = \frac{1}{2m}k^2$$

$$\rho_1 = i(\psi - \psi^+), \qquad j_1 = \frac{1}{2m}\nabla(\psi + \psi^+)$$

$$\rho_2 = (\psi + \psi^+), \qquad j_1 = \frac{1}{2mi}\nabla(\psi - \psi^+)$$

$$\rho_3 = \psi^+\psi, \qquad j_3 = \frac{1}{2mi}\psi^+\nabla\psi$$

$$[Q_3, Q_1] = -iQ_2$$

$$[Q_3, Q_2] = +iQ_1$$

$$[Q_1, Q_2] = 2\,iV$$

$$\langle 0|\rho_a(x)|k\rangle \sim \frac{1}{\sqrt{V}}e^{ik\cdot x}$$

$$\langle 0|j_a(x)|k\rangle \sim \frac{k}{2m\sqrt{V}}e^{ik\cdot x}$$

A single Goldstone boson couples to two currents. This is similar to what happens with magnons in Heisenberg ferromagnets. We have broken the relativistic necessity of having particles and antiparticles.

Goldstone II

Chadha-Nielsen Theorem

Given a Lagrangian invariant under a symmetry group G of dimension n, we can construct the corresponding Noether charges and fields satisfying:

$$\det\langle 0|[\Phi_i, Q_a]|0\rangle \neq 0, \qquad i, a = 1, \dots m$$

For local operators A,B,

$$x \to \infty: \ |\langle 0|[A(x,t), B(0)]|0\rangle| \to e^{-\tau|x|}, \qquad \tau > 0$$

Translational invariance is maintained, then:

$$n_I + 2n_{II} = m$$

$$E_I \sim p_I^{2r+1} \qquad E_{II} \sim p_{II}^{2r}$$

Chadha and Nielsen proved an inequality, many other people have worked on this problem, including Guralnik, Hagen, Kibble, and more recently Leutwyler, Murayama, Brauner, Watanabe, Nicolis, Piazza…

$$n_{\rm BG} - n_{\rm NGB} = \frac{1}{2}{\rm rank}\,\rho,$$

$$\rho_{ij} \equiv \lim_{\Omega\to\infty} \frac{-i}{\Omega}\langle 0|[Q_i, Q_j]|0\rangle,$$

Goldstone III

We can consider a different situation (Nicolis-Piazza…), spontaneous symmetry probing: This is a controlled way of breaking Lorentz invariance, at the same time that you break a symmetry (SSP). Hence we explore a class of theories which are secretly relativistic, but whose low energy behavior in the situation considered breaks Lorentz invariance spontaneously. Hence some version of the Chadha-Nielsen theorem applies.

This happens in our case of interest which will explores a QFT with a global symmetry group in the sector of large values of some of the charges, or charge densities. Similar situations arise in the study of QCD at finite baryon density, non-topological solitons, Q-balls…

We find that in the study of "natural hamiltonians", where there are no obvious expansion parameters, we can get access to some information by considering properties of the system at large values of the charge, and obtain a systematic Expansion in powers of 1/Q. Similar to the case of large-N

Ground state + Goldstones + 1/Q corrections

Some of these contributions are universal

Including charge density

Consider the simplest case of a classical theory with Hamiltonian H and charge Q. Since we will be considering the case where translations and rotations are maintained, and we work at finite volume, let's be sloppy and use Q and charge density interchangeably ([H,Q]=0), the charge generates canonically the symmetry transformations in the theory. Imposing the constraint of constant charge density should be done systematically

$$\int \rho \, \mathrm{d}x = \text{vol.} \times \bar{\rho} = \overline{Q}\,.$$

$$\dot{p}_i = \{p_i, H\}, \qquad \dot{q}_i = \{q_i, H\},$$

$$\dot{\chi} = \{\chi, H\}, \qquad \dot{\rho} = \{\rho, H\} = 0,$$

$$\{\chi, Q\} = 1\,,$$

$$H = \frac{1}{2}\sum_{k=0}^{N} f_k(q) p_k^2 + \frac{1}{2}\sum_{k=0}^{N} g_k(q)(\nabla q_k)^2 + V(q),$$

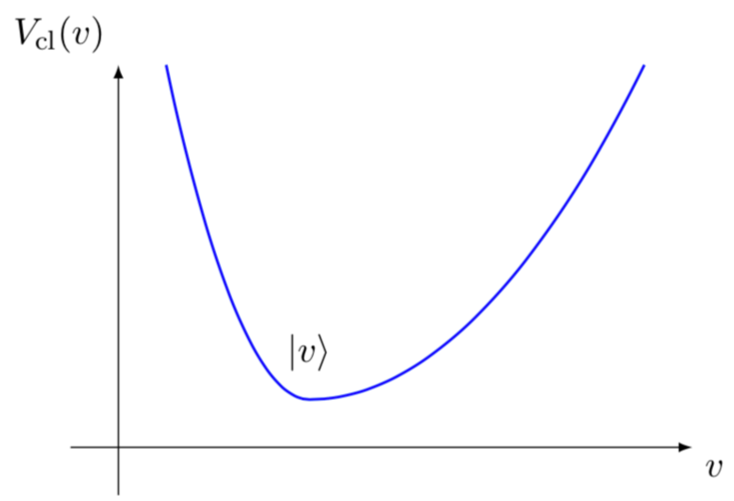
$$\delta_\epsilon \chi = \epsilon\,,$$

$$\dot{p}_i = \partial_i f_0\, \bar{\rho}^2 + \partial_i V = 0\,,$$

$$\dot{q}_i = 0\,,$$

$$\dot{\chi} = f_0(q_i)\bar{\rho}\,.$$

$$p_i = 0\,, \qquad q_i = \bar{q}_i(\bar{\rho})\,, \qquad \chi = f_0(\bar{q}_i(\bar{\rho}))\bar{\rho} t = \mu(\bar{\rho}) t$$



Quantum mechanically

As in the interpretation of effective actions, we can use the variational approach. We want to find the state minimizing:

$$\langle v|H|v\rangle \qquad \langle v|v\rangle = 1, \qquad \langle v|\rho|v\rangle = \bar{\rho}$$

$$\langle v|H - E_0 - m\rho|v\rangle$$

$$(H - E_0 - m\rho)|v\rangle = 0$$

$$\langle v|\dot{\chi}|v\rangle = \mu \qquad\qquad \mathcal{H} = H - \mu\rho - E_0$$

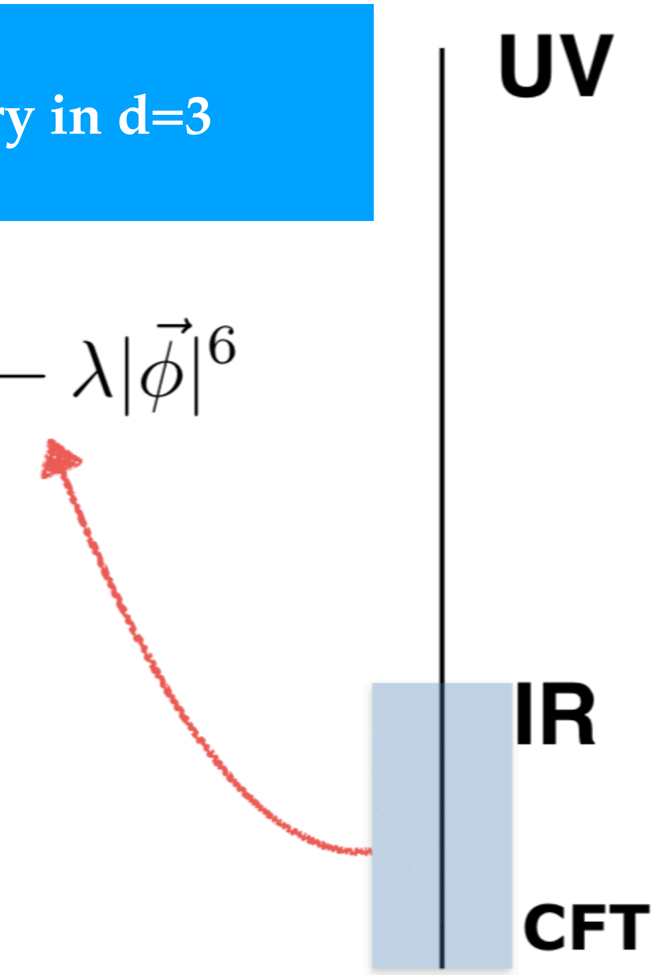
In this general context one can apply the standard techniques in Goldstone's theorem to show the possible existence of gapless excitations. It is important to notice that we are using the time evolution with the standard Hamiltonian. We find simpler to work an example.

$$(H - \mu Q)|v\rangle = 0$$

O(2n) model

We will consider later a naive scale invariant theory in d=3

$$\mathcal{L} = \partial_\mu \vec{\phi}^\dagger \partial^\mu \vec{\phi} - \frac{\mathcal{R}}{8} |\vec{\phi}|^2 - \lambda |\vec{\phi}|^6$$



- complex scalar $\vec{\phi} : \mathbb{R}_t \times S^2(r_0) \to \mathbb{C}^n$
- $\lambda \sim \mathcal{O}(1)$ is a Wilsonian parameter
- $\mathcal{R} = 2/r_0^2$ is the Ricci scalar
- The scale-invariant action enjoys in real DOFs a global $O(2n)$ symmetry

- Fix $k \leq n$ charges Q_i in the Cartan sub-algebra of $O(2n)$

- The analysis of the classical ground state goes in parallel to that of the $O(2)$ vacuum

- In particular, the only non-vanishing amplitude is fixed by the centrifugal potential such that $v \sim \mathcal{O}(Q^{1/4})$

- Classical ground-state: by $U(n)$ symmetry can be written as

$$\langle \vec{\phi} \rangle = (\underbrace{0, \ldots, 0}_{k-1}, v\, \mathrm{e}^{i\mu t}, \underbrace{0, \ldots, 0}_{n-k})$$

Time-dependent!

with $\mu \sim \mathcal{O}(\sqrt{Q})$

- Fix $k \leq n$ charges Q_i in the Cartan sub-algebra of $O(2n)$

- Semi-classically we find
  - the relativistic Goldstone (from the Abelian subsector)

$$\omega_{\mathbf{p}} = \frac{|\mathbf{p}|}{\sqrt{2}}$$

  - $k-1$ non-relativistic Goldstones

$$\omega_{\mathbf{p}} = \frac{|\mathbf{p}|^2}{2\mu}$$

  - all other radial modes and spectators are μ-massive

Computing anomalous dimensions

$$D(Q) = \frac{c_{3/2}}{\sqrt{4\pi}} Q^{3/2} + \sqrt{4\pi}\, c_{1/2}\, Q^{1/2} - 0.0937256 + (Q^{-1/2})$$

$(r_0 = 1)$

- The energy on $S^2(r_0)$ is dictated by the condensate v and μ and the vacuum energy of the relativistic Goldstone
- The effect of non-relativistic Goldstones and any quantum loops is suppressed

$$\langle \vec{\phi} \rangle = (\underbrace{0, \ldots, 0}_{k-1}, v\, e^{i\mu t}, \underbrace{0, \ldots, 0}_{n-k})$$

$$\omega_{\mathbf{p}} = \frac{|\mathbf{p}|}{\sqrt{2}}$$

Computing anomalous dimensions

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➡ universal prediction for all $O(n)$

- There is no Q^0 contribution from the classical ground state $|Q\rangle$ to the energy
- Only the relativistic Goldstone contributes via its universal dispersion

$$\omega_{\mathbf{p}} = \frac{|\mathbf{p}|}{\sqrt{2}}$$

Computing anomalous dimensions

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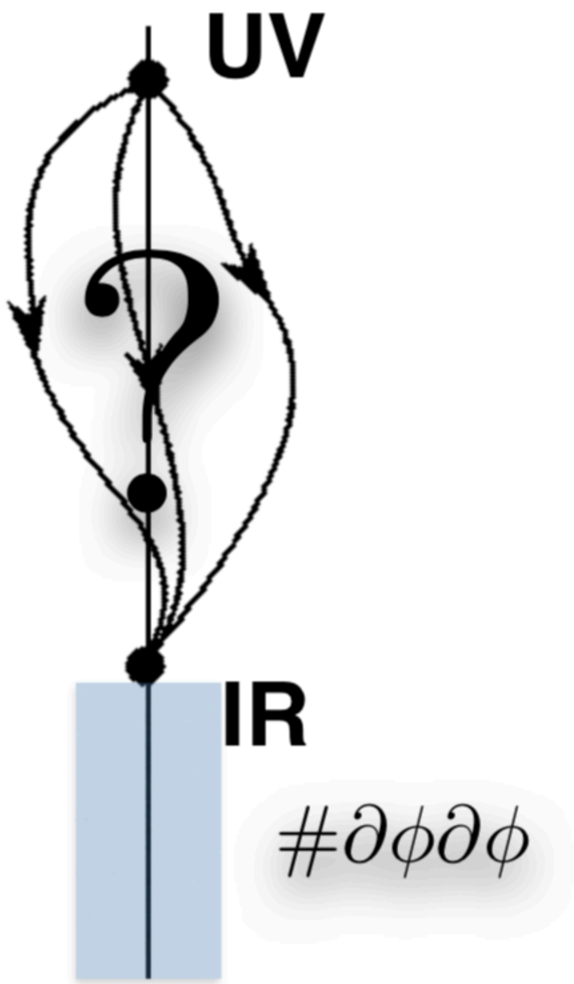
- $\mathcal{O}(1)$ ignorance coefficients
- depend on the IR coupling(s) and the specifics of the RG flow through

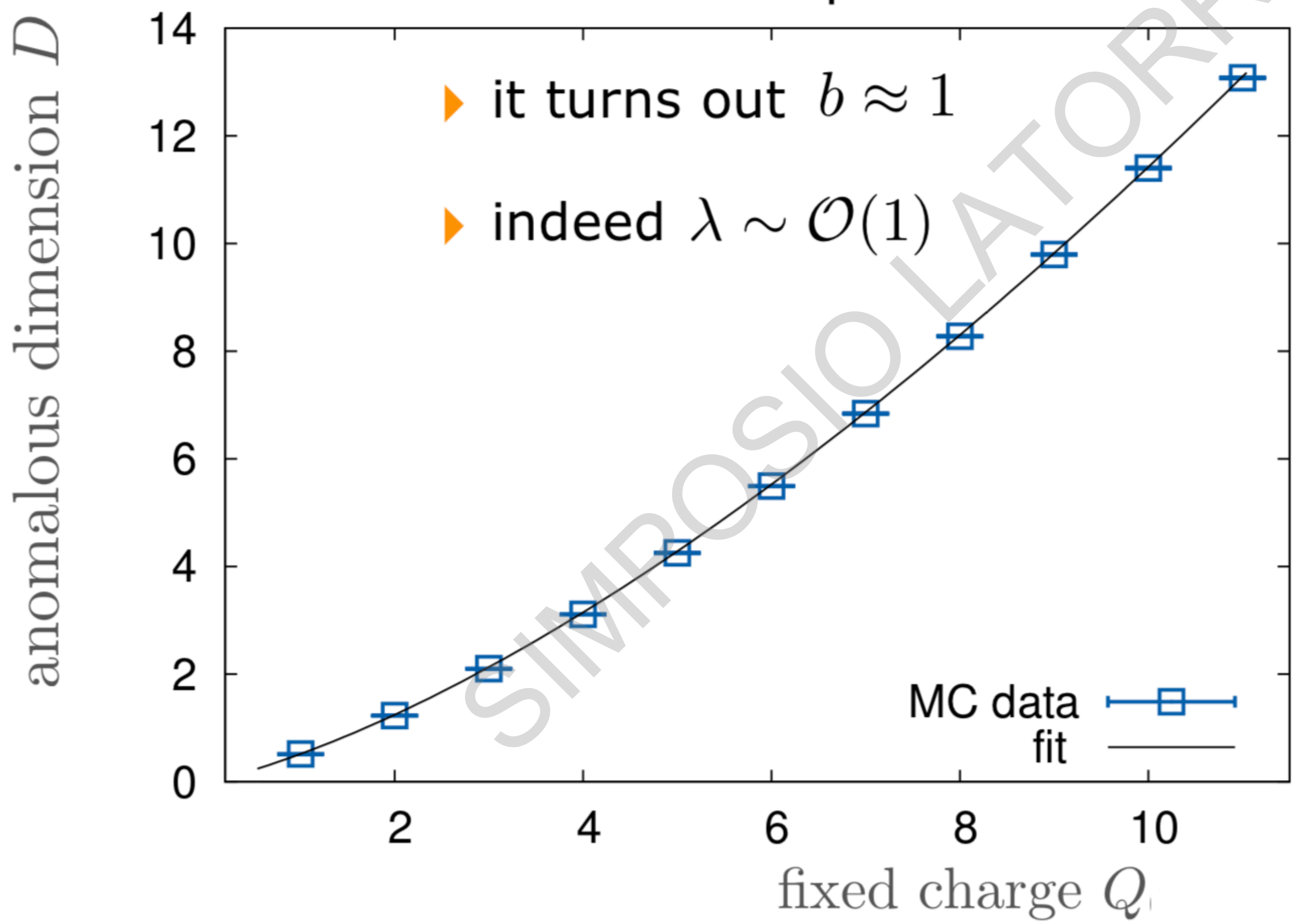


$\longleftrightarrow$ recall from the Abelian case: two undetermined parameters λ and b

Computing anomalous dimensions

$$D(Q) = \frac{c_{3/2}}{\sqrt{4\pi}} Q^{3/2} + \sqrt{4\pi}\, c_{1/2}\, Q^{1/2} - 0.0937256 + (Q^{-1/2})$$

- fix them via parameter-fit to lattice data
- it turns out $b \approx 1$
- indeed $\lambda \sim \mathcal{O}(1)$



scalar $O(2)$

at fixed charge

universal number:
0.09 ± 0.05
seen with uncertainty

Analytic vs Lattice

[Banerjee,Chandrasekharan, Orlando'2017]

$$c_{3/2}\, c_{1/2} \approx \frac{1}{12}$$

In the naive model at tree level, and in the fit, it seems that this relation is well satisfied !?!

Computing anomalous dimensions

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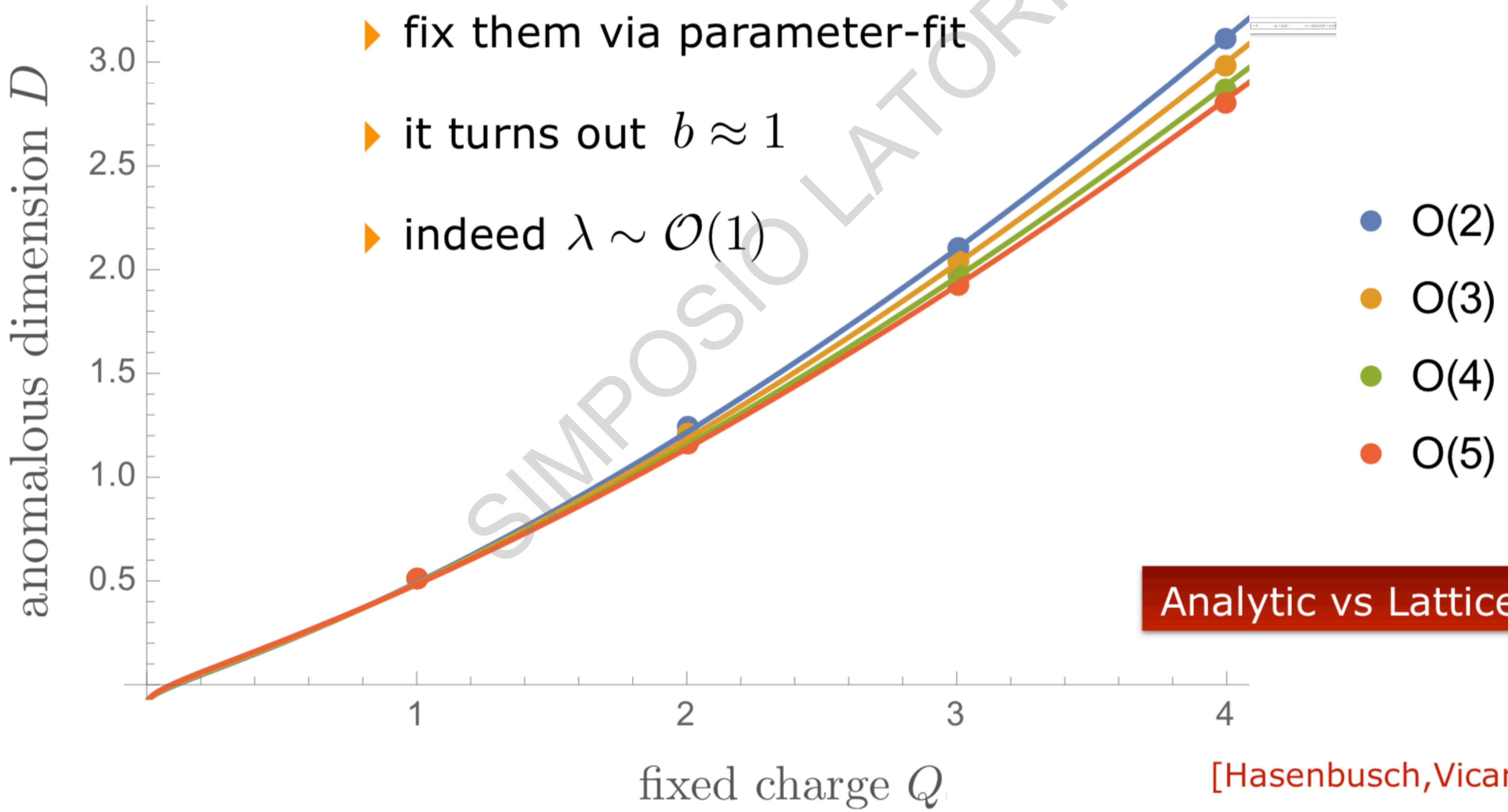

Analytic vs Lattice

[Hasenbusch,Vicari’2011]

Feliz cumpleaños old friend!

Gory details 1

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\phi^a\,\partial^\mu\phi^a - \frac{1}{2}V(\phi^a\phi^a), \qquad \mathrm{U}(n) \subset O(2n),$$

$$\varphi_1 = \frac{1}{\sqrt{2}}\,(\phi_1 + i\phi_2)\;, \qquad \varphi_2 = \frac{1}{\sqrt{2}}\,(\phi_3 + i\phi_4)\;, \qquad \int \mathrm{d}^{d-1}x\,\rho_i = \overline{Q}_i = \text{vol.} \times \bar{\rho}_i\;,$$

$$\begin{cases} \varphi_i = \dfrac{1}{\sqrt{2}}A_i\,e^{i\mu t}, & i = 1,\dots,k\,, \\ \varphi_{k+j} = 0, & j = 1,\dots,n-k\,, \end{cases} \qquad \begin{aligned} \bar{\rho}_i &= A_i^2\sqrt{V'(A_1^2+\dots+A_k^2)}\,, \\ \mu &= \sqrt{V'(A_1^2+\dots+A_k^2)}\,. \end{aligned}$$

Same for all fields

$$H - \mu(\rho_1 + \rho_2 + \dots + \rho_k)\,, \qquad v^2 = \sum_{i=1}^{k} A_i^2 = \frac{1}{\mu}\sum_{i=1}^{k}\bar{\rho}_i = \frac{\bar{\rho}}{\mu}\,.$$

$$\mu = \mathcal{O}\left(\rho^{1/(d-1)}\right) \qquad \text{and} \qquad v = \mathcal{O}\left(\rho^{(d-2)/(2(d-1))}\right).$$

Gory details 2

$$\mathcal{L}_\mu = \sum_{k=1}^{k}(\partial_t - i\mu)\varphi_i^*(\partial_t + i\mu)\varphi_i + \sum_{i=k+1}^{n}\dot\varphi_i^*\dot\varphi_i - \sum_{k=1}^{n}\nabla\varphi_i^*\nabla\varphi_i - V(2|\varphi_1|^2 + \cdots + 2|\varphi_n|^2)\,.$$

$$i\mu\sum_{i=1}^{k}(\dot\varphi_i^*\varphi_i - \varphi_i^*\dot\varphi_i) = i\mu(\dot{\vec\varphi}^\dagger\cdot\vec\varphi - \vec\varphi^\dagger\cdot\dot{\vec\varphi})\,,$$

$$\begin{cases}\langle\varphi_i\rangle = \frac{1}{\sqrt2}A_i, & i = 1,\ldots,k\,,\\ \langle\varphi_i\rangle = 0, & i = k+1,\ldots,n\,.\end{cases}$$

Using the symmetries we can rotate the ground state to a rather simple configuration

$$M\,\langle\vec\varphi\rangle = \Big(\underbrace{0\,,\ldots,\,0}_{k-1},\,\frac{v}{\sqrt2}\,,\underbrace{0\,,\ldots,\,0}_{n-k}\Big)\,.$$

U(k-1) sector **O(2) sector**

$$\varphi_k = \frac{1}{\sqrt2}e^{i\mu t + i\hat\phi_{2k}/v}\left(v + \hat\phi_{2k-1}\right),$$

$$\begin{cases}\hat\phi_{2k-1}\to\hat\phi_{2k-1}\\ \hat\phi_{2k}\to\hat\phi_{2k}+\theta\,,\end{cases}$$

$$\varphi_i = e^{i\mu t}\hat\varphi_i\,,$$

$$\mathcal{L} = \sum_{i=1}^{k-1}(\partial_t - i\mu)\varphi_i^*(\partial_t + i\mu)\varphi_i + \frac12\dot\phi_{2k-1}\dot\phi_{2k-1} + \sum_{i=k+1}^{n}\dot\varphi_i^*\dot\varphi_i + \frac12(v+\phi_{2k-1})^2\left(\left(\mu + \frac{\dot\phi_{2k}}{v}\right)^2 - \frac{(\nabla\phi_{2k})^2}{v^2}\right) - \sum_{i=1}^{n-1}\nabla\varphi_i^*\nabla\varphi_i - \frac12(\nabla\phi_{2k-1})^2 - \frac12 V\left(2|\varphi_1|^2 + \cdots + 2|\varphi_{k-1}|^2 + |v+\phi_{2k-1}|^2 + 2|\varphi_{k+1}|^2 + \cdots + 2|\varphi_n|^2\right),$$

Gory details 2

$$V''(v^2) = \frac{2c^2}{1-c^2}\frac{\mu^2}{v^2}\,.$$

$$\omega^2 = \left(-\mu + \sqrt{p^2+\mu^2}\right)^2 = \frac{p^4}{4\mu^2} - \frac{p^6}{8\mu^4} + \mathcal{O}(\mu^{-6}) \qquad k-1 \text{ times}$$

$$\omega^2 = \left(\mu + \sqrt{p^2+\mu^2}\right)^2 = 4\mu^2 + 2p^2 + \mathcal{O}(\mu^{-2}) \qquad k-1 \text{ times}$$

$$\omega_-^2 = c^2p^2 + \frac{\left(1-c^2\right)^3 p^4}{4\mu^2} + \mathcal{O}(\mu^{-4}) \qquad \text{one time}$$

$$\omega_+^2 = \frac{4\mu^2}{1-c^2} + (2-c^2)p^2 + \mathcal{O}(\mu^{-2}) \qquad \text{one time.}$$

A very technical analysis shows that if we integrate out the massive modes, the effective couplings have the form:

$$\frac{\lambda^{i_1\ldots i_m}}{\mu^{-d+m/2(d-1)}} = \frac{\lambda^{i_1\ldots i_m}}{\mu^{\Omega_m}}\,. \qquad \frac{\lambda^{i_1\ldots i_m}}{\bar{\rho}^{(m/2-d/(d-1))}}\,.$$

$$\Omega_m = \frac{m}{2}(d-1) - d > 0 \qquad \text{for } m \geq 4$$

Summary

The different avatars of symmetry realizations in QFT continue to amaze us. Noether's paper was published in 1918. Her fundamental vision, when combined with quantum mechanics has provided a vast landscape that has not yet yielded all its secrets…

The hybrid interplay between R and NR d.o.f. in a relativistic microscopic theory may yield interesting insights in the study of CFTs in regions that could be difficult to reach using bootstrap approaches. The argument is also independent of dimension, and it can be combined with other tools, like large-N expansions.

Direct approaches in the bootstrap context have already started (Jafferis, Zhiboedov, Caron-Huot,…)

Thank you