

# Quantum Primes

Workshop “Latorre Fest”

Barcelona, 31 May 2019



Germán Sierra

Instituto de Física Teórica UAM-CSIC, Madrid

**Querido José Ignacio**  
**Felices 3 x 4 x 5 's**



Where this story begun



Work done in collaboration with

José Ignacio Latorre

U. Barcelona,  
BSC



Diego García Martín

U. Barcelona  
IFT, Madrid



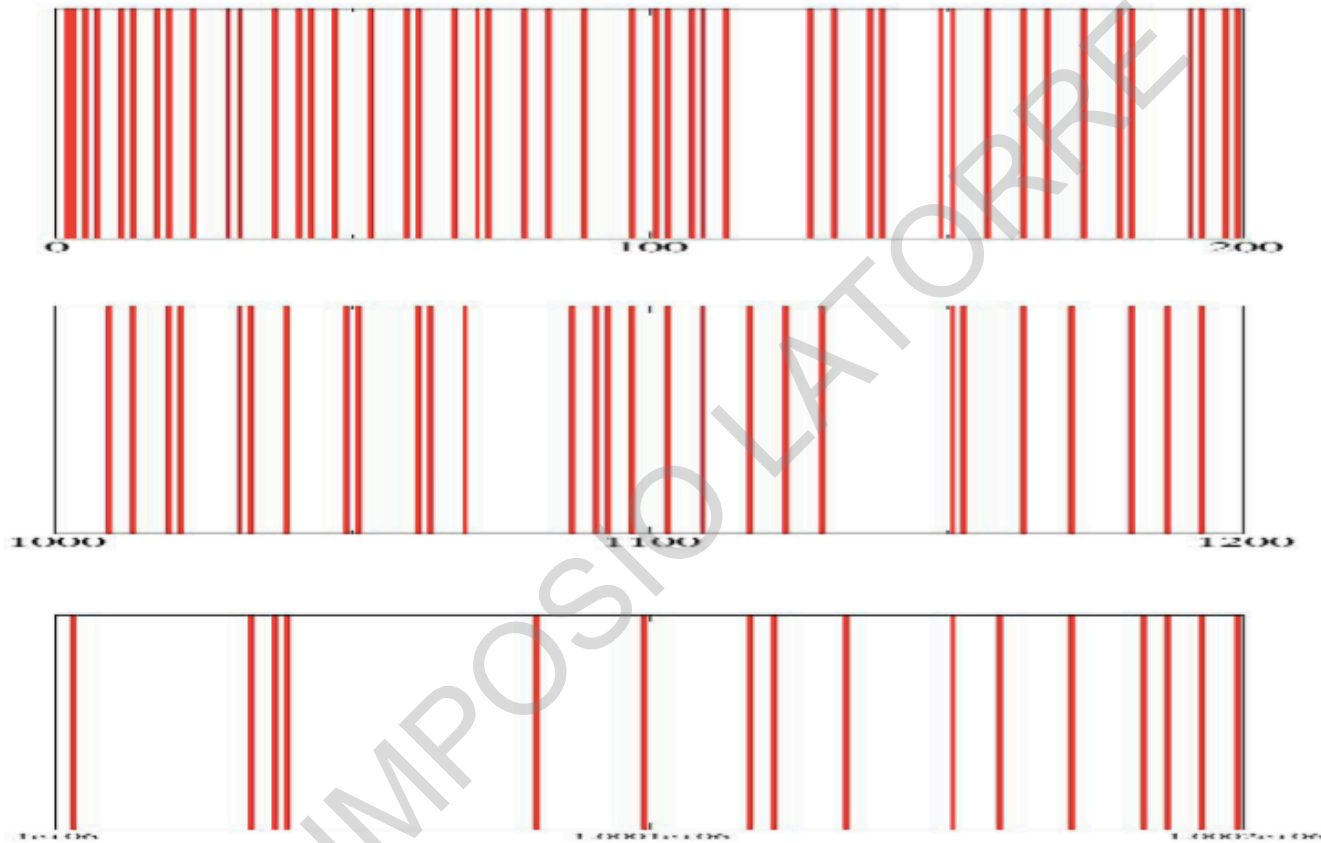
# Outline

- **Part I:** Prime Numbers and Riemann Zeros
- **Part II:** Random Matrix Theory and Quantum Chaos
- **Part III:** The Prime State

# Part I

SIMPOSIO LATORRE

# Order and chaos in the primes



## Order, Order !! Gauss Law

The number of primes less than  $N$  is approximately

$$\pi(N) \sim \frac{N}{\ln N}, \quad N \rightarrow \infty$$



$N$	$\pi(N)$	$\frac{N}{\ln N}$
1000	168	145
10000	1229	1086
100000	9592	8686
1000000	78498	72382
10000000	664579	620420
100000000	5761455	5428681



## Gauss Law = Prime Number Theorem

(Hadamard, de la Vallee-Poussin, 1896)

$$\lim_{N \rightarrow \infty} \frac{\pi(N)}{Li(N)} = 1$$

Logarithmic integral function

$$Li(x) = \int_2^x \frac{dt}{\ln t} = \frac{x}{\ln x} + \frac{x}{(\ln x)^2} + \dots$$

Density of primes  $\frac{d\pi(x)}{dx} \sim \frac{1}{\ln x}$

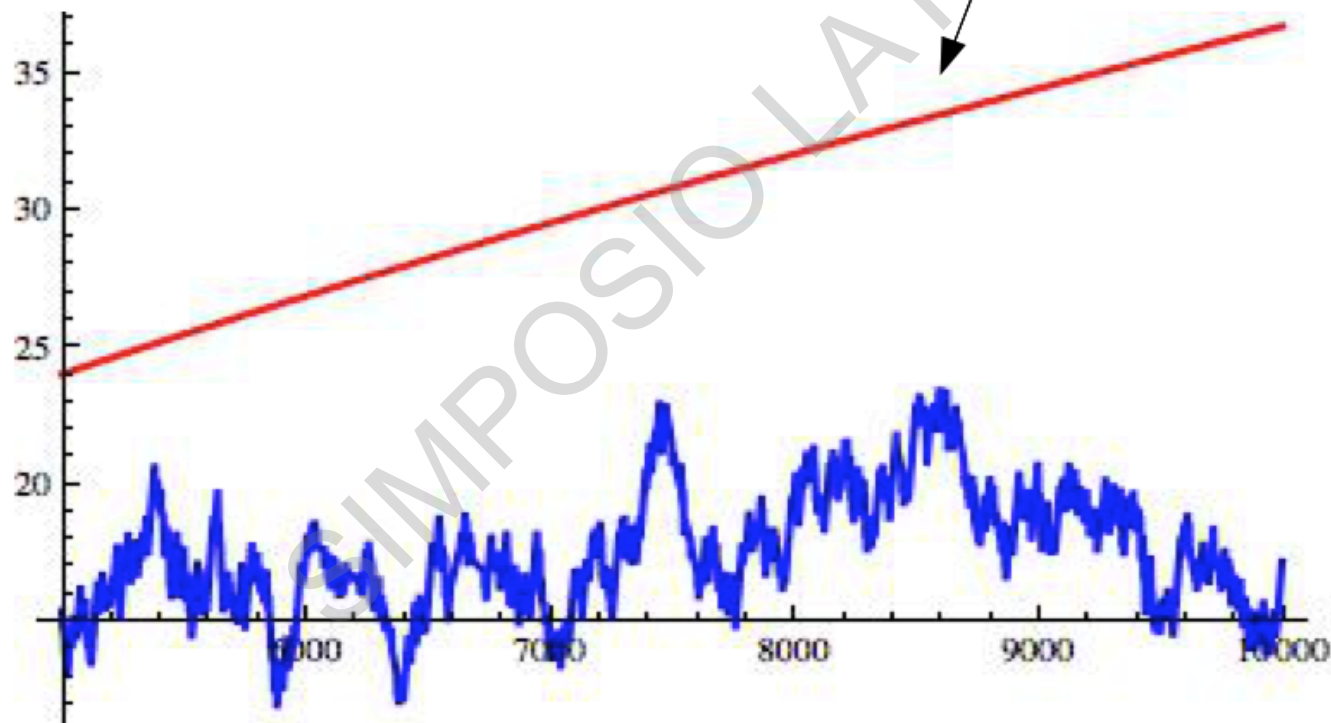
$x$	$\pi(x)$	$\pi(x) - x / \ln x$	$\text{li}(x) - \pi(x)$
10	4	-0.3	2.2
$10^2$	25	3.3	5.1
$10^3$	168	23	10
$10^4$	1,229	143	17
$10^5$	9,592	906	38
$10^6$	78,498	6,116	130
$10^7$	664,579	44,158	339
$10^8$	5,761,455	332,774	754
$10^9$	50,847,534	2,592,592	1,701
$10^{10}$	455,052,511	20,758,029	3,104
$10^{11}$	4,118,054,813	169,923,159	11,588
$10^{12}$	37,607,912,018	1,416,705,193	38,263
$10^{13}$	346,065,536,839	11,992,858,452	108,971
$10^{14}$	3,204,941,750,802	102,838,308,636	314,890
$10^{15}$	29,844,570,422,669	891,604,962,452	1,052,619
$10^{16}$	279,238,341,033,925	7,804,289,844,393	3,214,632
$10^{17}$	2,623,557,157,654,233	68,883,734,693,281	7,956,589
$10^{18}$	24,739,954,287,740,860	612,483,070,893,536	21,949,555
$10^{19}$	234,057,667,276,344,607	5,481,624,169,369,960	99,877,775
$10^{20}$	2,220,819,602,560,918,840	49,347,193,044,659,701	222,744,644
$10^{21}$	21,127,269,486,018,731,928	446,579,871,578,168,707	597,394,254
$10^{22}$	201,467,286,689,315,906,290	4,060,704,006,019,620,994	1,932,355,208
$10^{23}$	1,925,320,391,606,803,968,923	37,083,513,766,578,631,309	7,250,186,216
$10^{24}$	18,435,599,767,349,200,867,866	339,996,354,713,708,049,069	17,146,907,278
$10^{25}$	176,846,309,399,143,769,411,680	3,128,516,637,843,038,351,228	55,160,980,939
$10^{26}$	1,699,246,750,872,437,141,327,603	28,883,358,936,853,188,823,261	155,891,678,121
$10^{27}$	16,352,460,426,841,680,446,427,399	267,479,615,610,131,274,163,365	508,666,658,006

Wikipedia

## Chaos in the primes

The fluctuations of  $\pi(x)$  around  $Li(x)$  are expected to be bounded by

$$|Li(x) - \pi(x)| < \frac{1}{8\pi} \sqrt{x \log x}$$



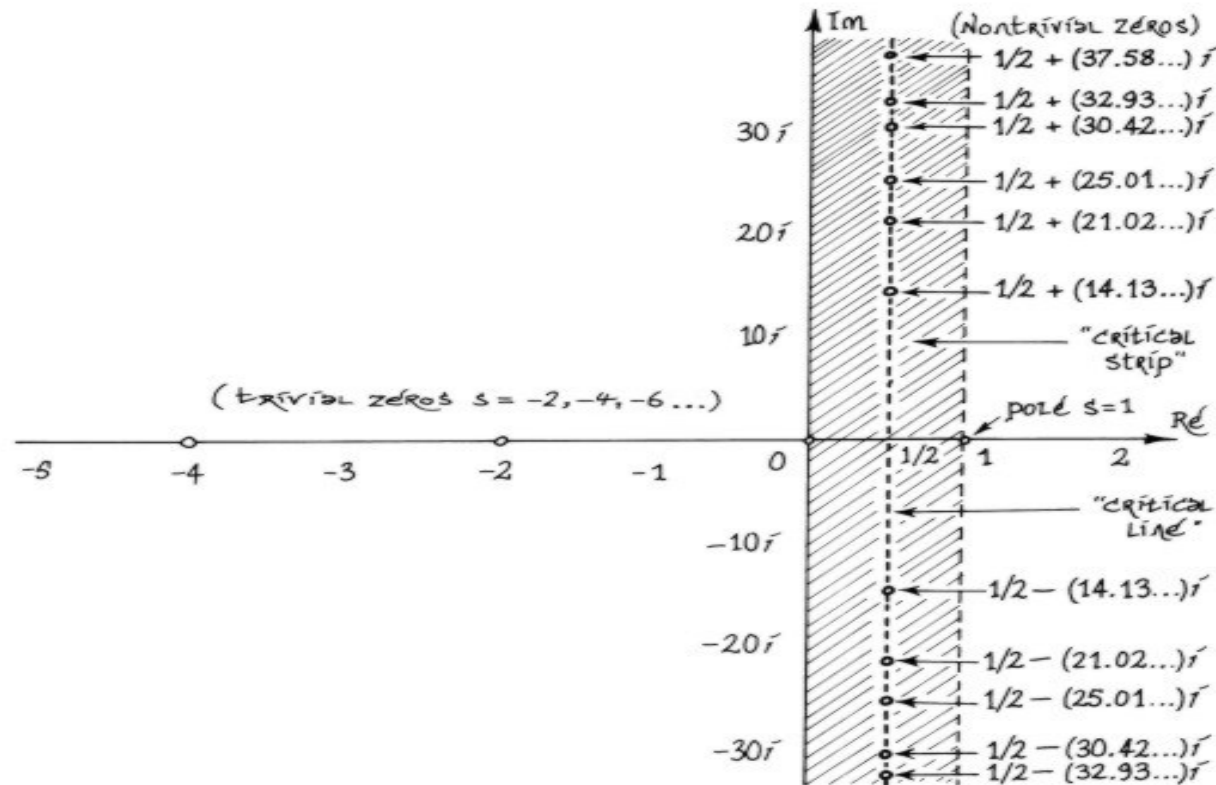
This statement is equivalent to the **Riemann hypothesis (RH)**

# Riemann Zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p=2,3,\dots} \frac{1}{1-p^{-s}} \quad \text{Euler formula}$$

$Re\ s > 1$

**RH: Non trivial zeros of  $\zeta(s)$  have real part equal to 1/2**



## Twin primes

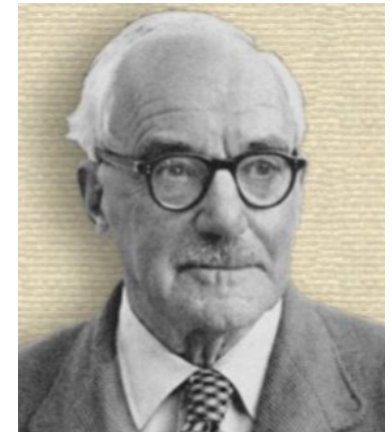
(3, 5), (5, 7), (11, 13), (17, 19), (29, 31), (41, 43), (59, 61), (71, 73), ....

$\pi_2(x)$  = number of primes  $p \leq x$  such that  $p + 2$  is also prime



Hardy - Littlewood conjecture (1923)

$$\pi_2(x) \sim 2 C_2 \frac{x}{(\ln x)^2}$$



Twin prime constant

$$C_2 = \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) = 0.6601618 \dots$$

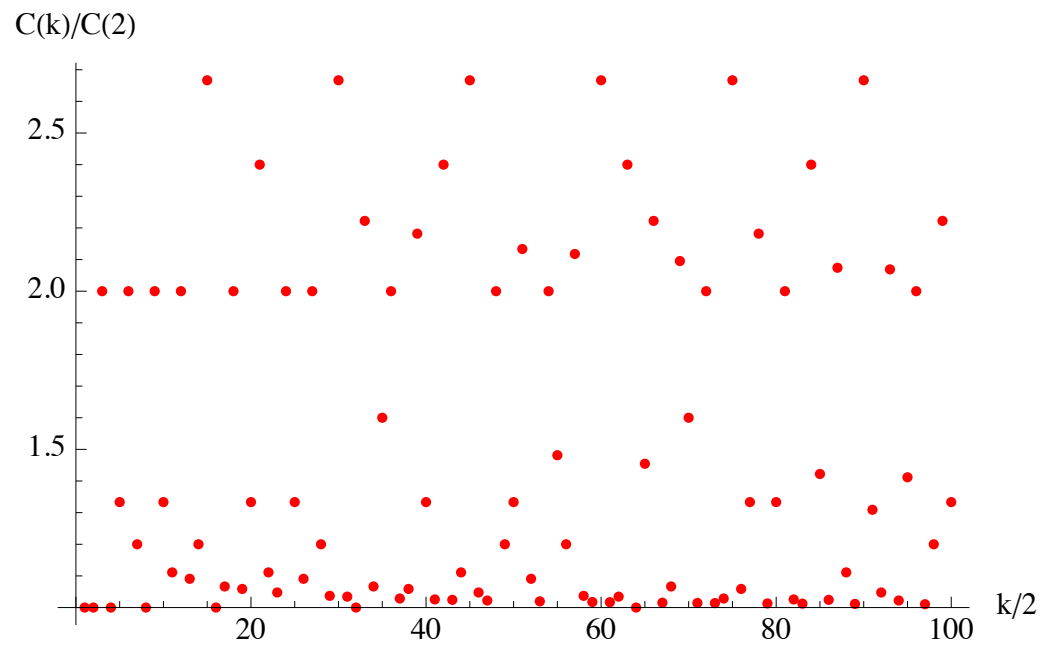
## Hardy-Littlewood conjecture

$\pi_2(k, x)$  = number of primes  $p \leq x$  such that  $p + k$  is also prime

$$\pi_2(k, x) \sim C(k) \frac{x}{(\ln x)^2}$$

$$C(k) = \begin{cases} 2 C_2 \prod_{p|k} \frac{p-1}{p-2} & \text{if } k : \text{even} \\ 0 & \text{if } k : \text{odd} \end{cases}$$

HL conjecture : pairwise correlations between the primes



Averaged form of the Hardy-Littlewood conjecture

$$\sum_{k=1}^K C(k) \sim K - \frac{1}{2} \log K, \quad K \gg 1$$

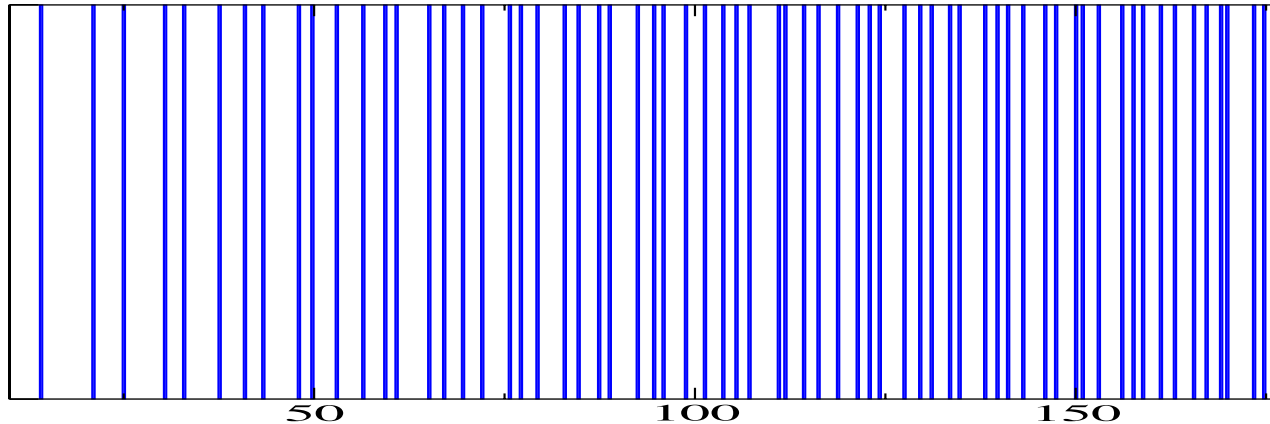
$$\langle C(k) \rangle \sim 1 - \frac{1}{2k} \quad (\text{Keating})$$

If primes were pairwise uncorrelated  $C(k) \sim 1$

# Part II



The Riemann zeros as spectrum  $\zeta\left(\frac{1}{2} + i E_n\right) = 0$



$$N_R(E) = \langle N(E) \rangle + N_{fl}(E)$$

Average  $\langle N(E) \rangle \approx \frac{E}{2\pi} \left( \log \frac{E}{2\pi} - 1 \right) + \frac{7}{8} + O(E^{-1})$

Fluctuation  $N_{fl}(E) = \frac{1}{\pi} \text{Arg} \zeta\left(\frac{1}{2} + i E\right) = O(\log E)$

## Pairwise correlations

Density of “zeros”

$$d(E) = \sum_n \delta(E - E_n)$$

Average density

$$\bar{d}(E) = \frac{d\langle N(E) \rangle}{dE} = \frac{1}{2\pi} \ln \frac{E}{2\pi}$$

Pair correlation

$$R_2(\varepsilon) = \langle d(E - \varepsilon/2)d(E + \varepsilon/2) \rangle$$

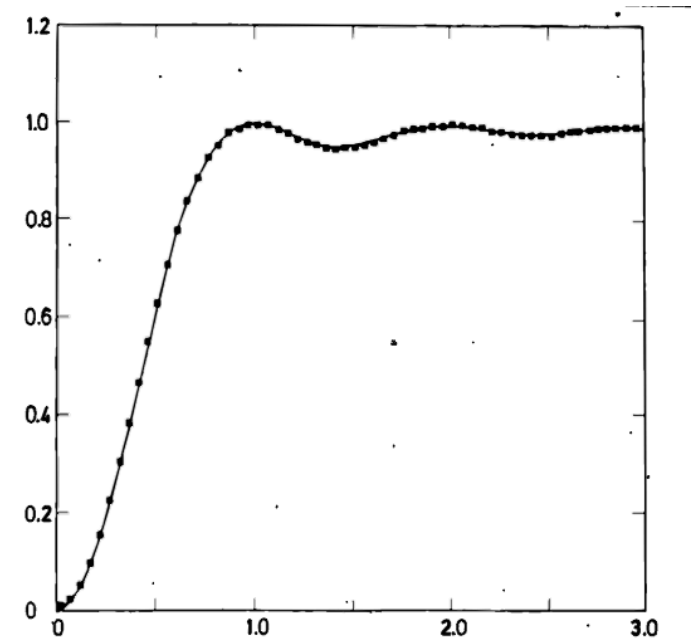
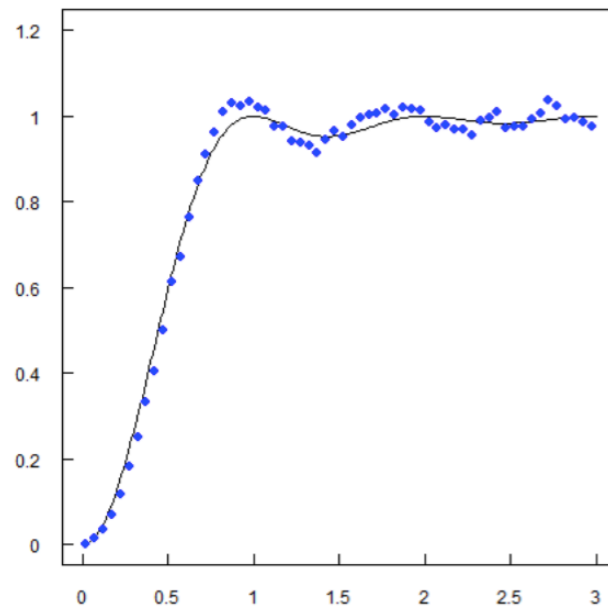
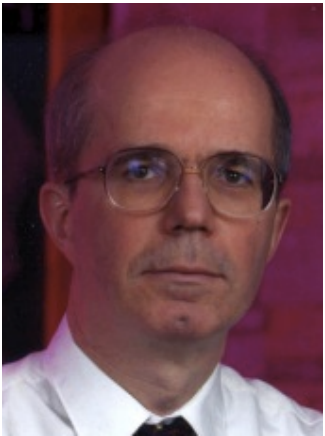


## Montgomery conjecture (1973)

In the limit  $E \rightarrow \infty$

$$R_2(\varepsilon) \rightarrow R_{GUE}(x) = 1 - \left( \frac{\sin \pi x}{\pi x} \right)^2 \quad x = \frac{\varepsilon}{\overline{d}(E)}$$

## Odlyzo numerics (1989)



## Prime Numbers

$$\pi_2(k, x)$$

Averaged HL conjecture

$$\langle C(k) \rangle \sim 1 - \frac{1}{2k}$$

Full HL conjecture

$$C(k)$$

## Riemann zeros

$$R_2(\varepsilon)$$

Montgomery conjecture

$$R_2(\varepsilon) \rightarrow R_{GUE}(x)$$

Berry, Bogomolny, Keating,...

$$R_2(\varepsilon) = R_{GUE}(x) + R_C(x)$$



Keating (1993)



Bogomolny, Keating (1996)

# TWIN PRIME CORRELATIONS FROM THE PAIR CORRELATION OF RIEMANN ZEROS

J. P. KEATING AND D. J. SMITH

arXiv:1903.07057v1

Full HL conjecture



$$R_2(\varepsilon) = R_{GUE}(x) + R_c(x)$$

Heuristic equivalence between the two conjectures

## Berry's analogy in Quantum Chaos (80's)



### Prime Numbers

Classical chaotic Hamiltonian

$p$  : primitive periodic orbit  
 $\log p$ : period

Breaks time reversal

Trace formula

### Riemann zeros

Quantum Hamiltonian

$E_n$  Eigenenergies

GUE statistics

Gutzwiller formula

**Spectral approach to prove the Riemann Hypothesis**

# Part III

## Quantum Computation and prime numbers (JI Latorre, GS, 2013 )

### Classical computer

n bits  $x = x_0 2^0 + x_1 2^1 + \dots + x_{n-1} 2^{n-1}, \quad x_i = 0,1, \quad x = 0,1,\dots,2^n - 1$

### Quantum computer

n qubits  $|x\rangle = |x_{n-1}, \dots, x_0\rangle = |x_{n-1}\rangle \otimes \dots \otimes |x_0\rangle$



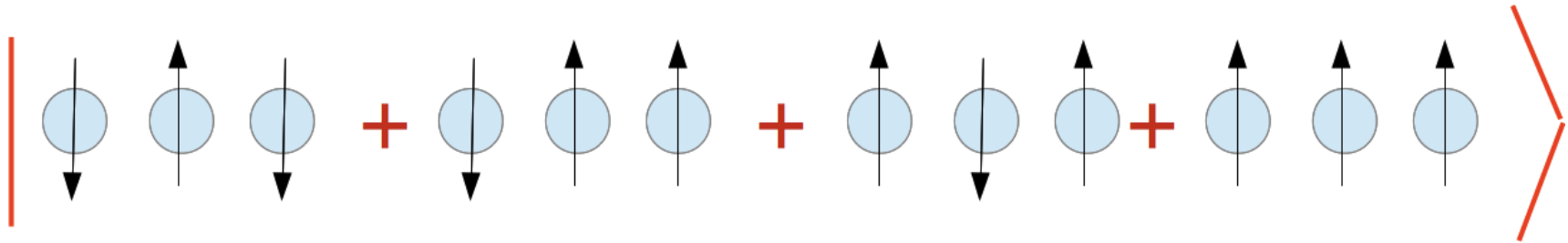
# The Prime State

$$|P(n)\rangle = \frac{1}{\sqrt{\pi(2^n)}} \sum_{p < 2^n \in \text{Primes}} |p\rangle$$

$\pi(2^n)$  is the prime counting function

Ex. n=3

$$|P(3)\rangle = \frac{1}{\sqrt{4}} (|2\rangle + |3\rangle + |5\rangle + |7\rangle)$$



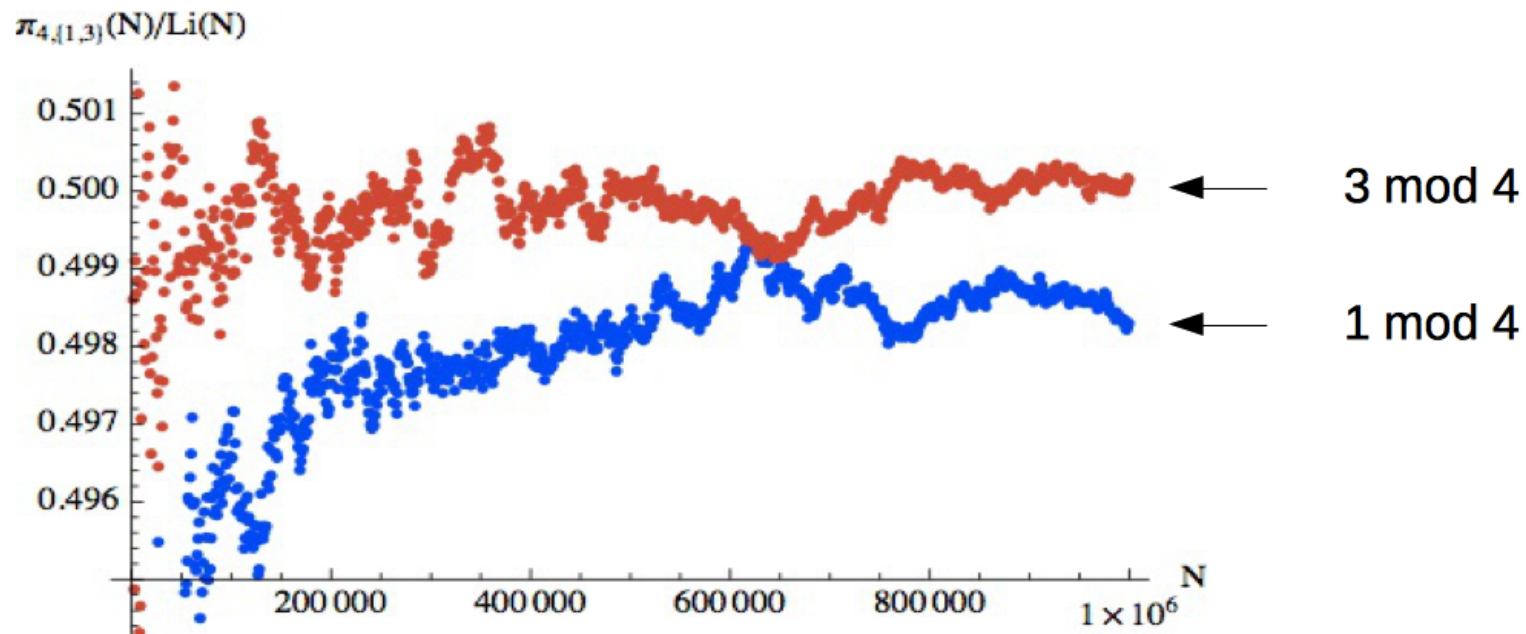
Efficient construction of the prime state:

Grover algorithm with an oracle that implements classical primality tests.

Quantum counting algorithm provides an estimate of  $\pi(2^n)$  that can be used to falsify the RH but not to prove it.

# Prime numbers in arithmetics progressions

The primes are equally distributed in the progressions 1 mod 4 and 3 mod 4

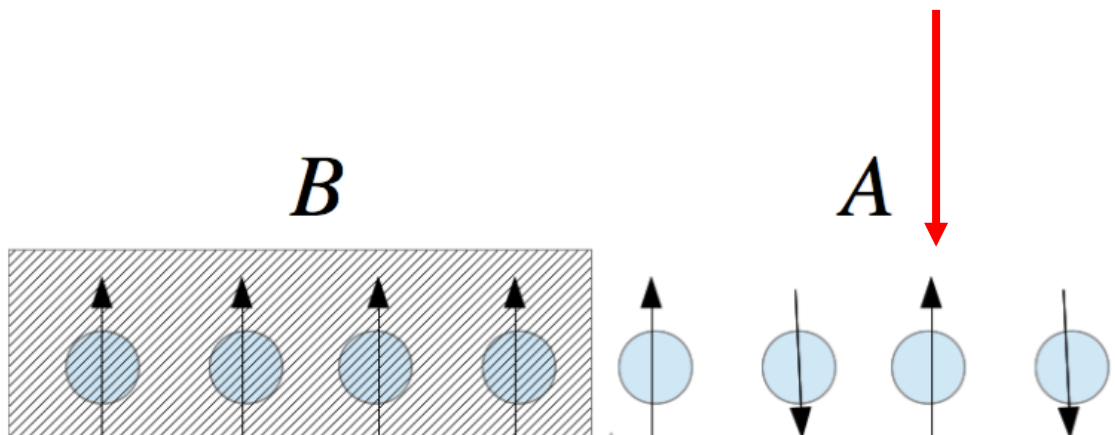


Prime number theorem for progressions  $\pi_{4,b}(x) \rightarrow \frac{x}{2 \ln x}, \quad b = 1,3$

# Chebyshev bias as magnetization

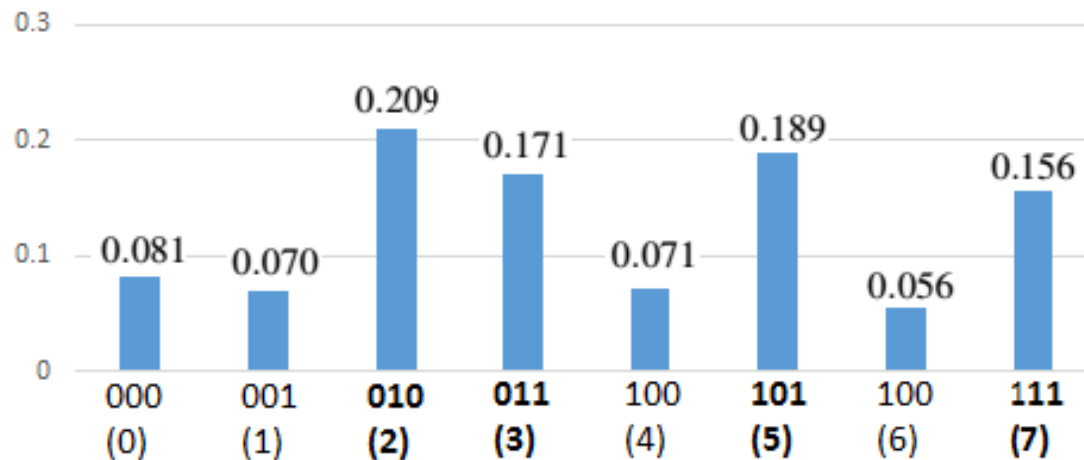
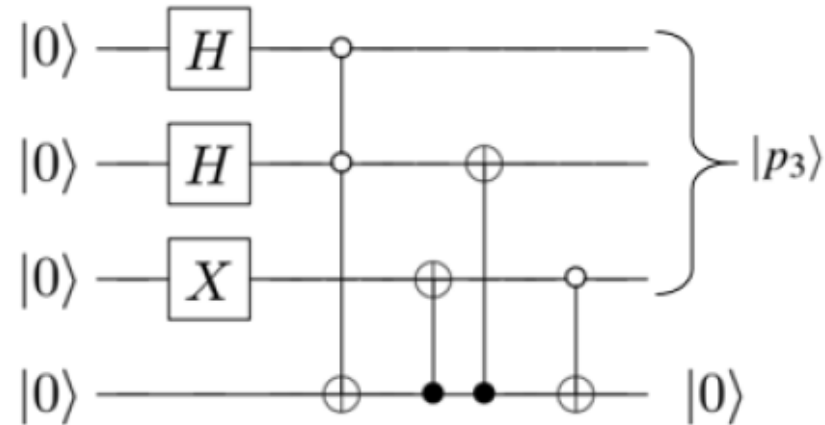
Chebyshev bias:  $\Delta(x) = \pi_{4,3}(x) - \pi_{4,1}(x)$

Magnetization of the qubit 1  $\langle \sigma_1^z \rangle = -\frac{\Delta(N)+1}{\pi(N)}, \quad N = 2^n$



# IBM quantum computer and the Prime state

(Diego García-Martín, GS, 2018)



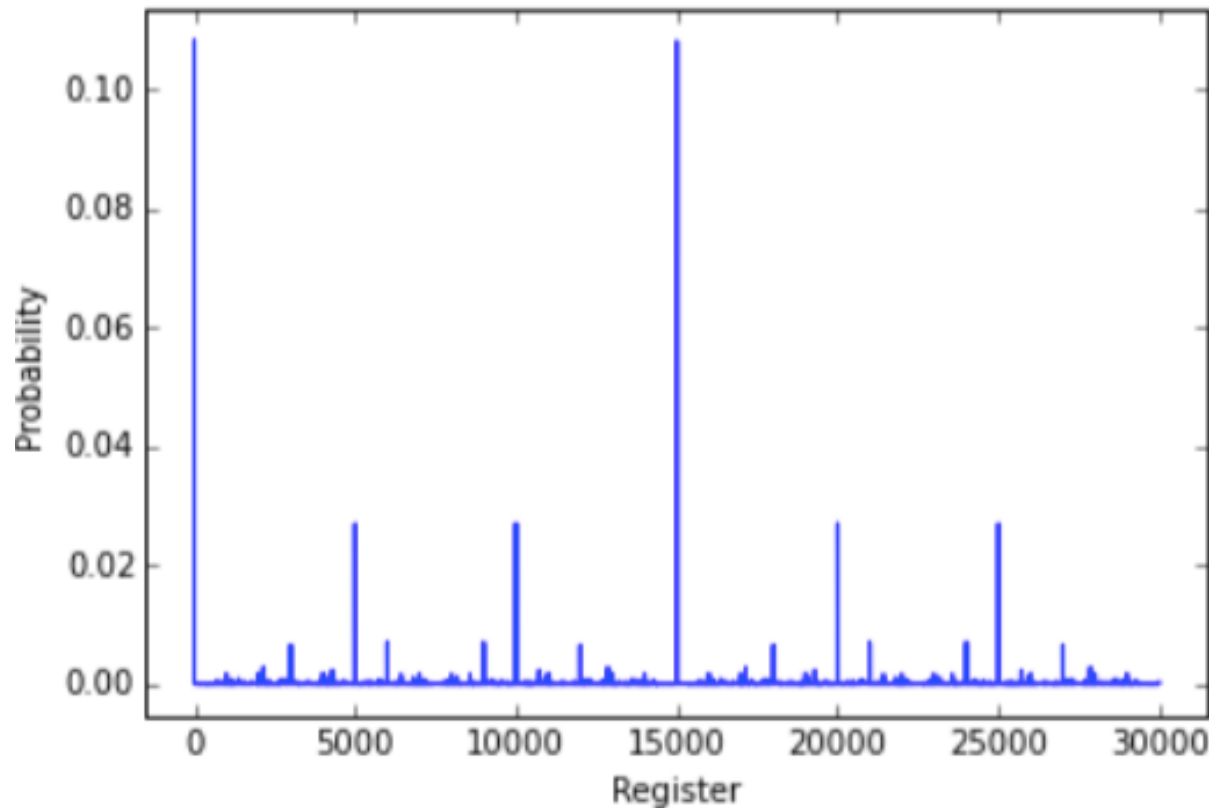
$$\langle \sigma_z^1 \rangle_{exp} = -0.183 \pm 0.029$$

$$\langle \sigma_z^1 \rangle_{th} = -0.500$$

# Quantum Fourier transform of the Prime state

Eduard Ribas Fernández , J.I. Latorre, TFG (18)

$$U_{QFT}|\mathbb{P}_N\rangle = \frac{1}{\sqrt{N\pi(N)}} \sum_{k=0}^{N-1} \left( \sum_{p \in \mathbb{P}_N} e^{\frac{2\pi i}{N} kp} \right) |k\rangle$$

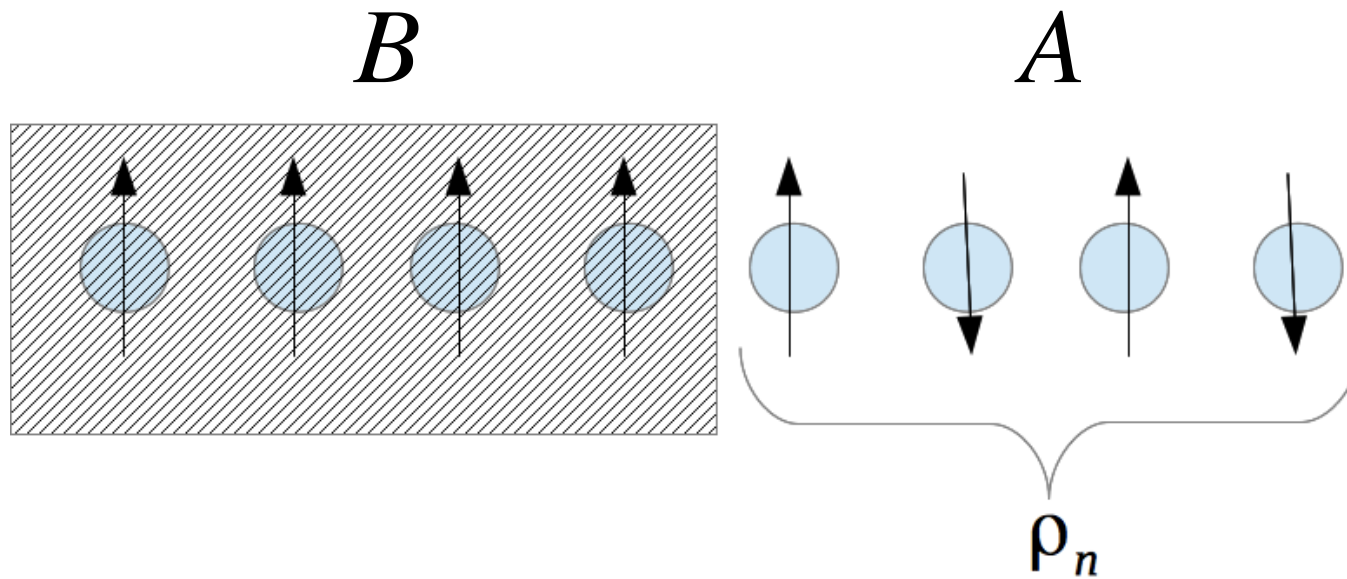


$$P(N) \propto \pi(N)^2$$
$$P\left(\frac{N}{2}\right) \propto \pi(N)^2 - 4\pi(N) + 4$$
$$P\left(\frac{N}{4}\right) \propto \Delta(N)^2 + 1$$

# Entanglement in the Prime state



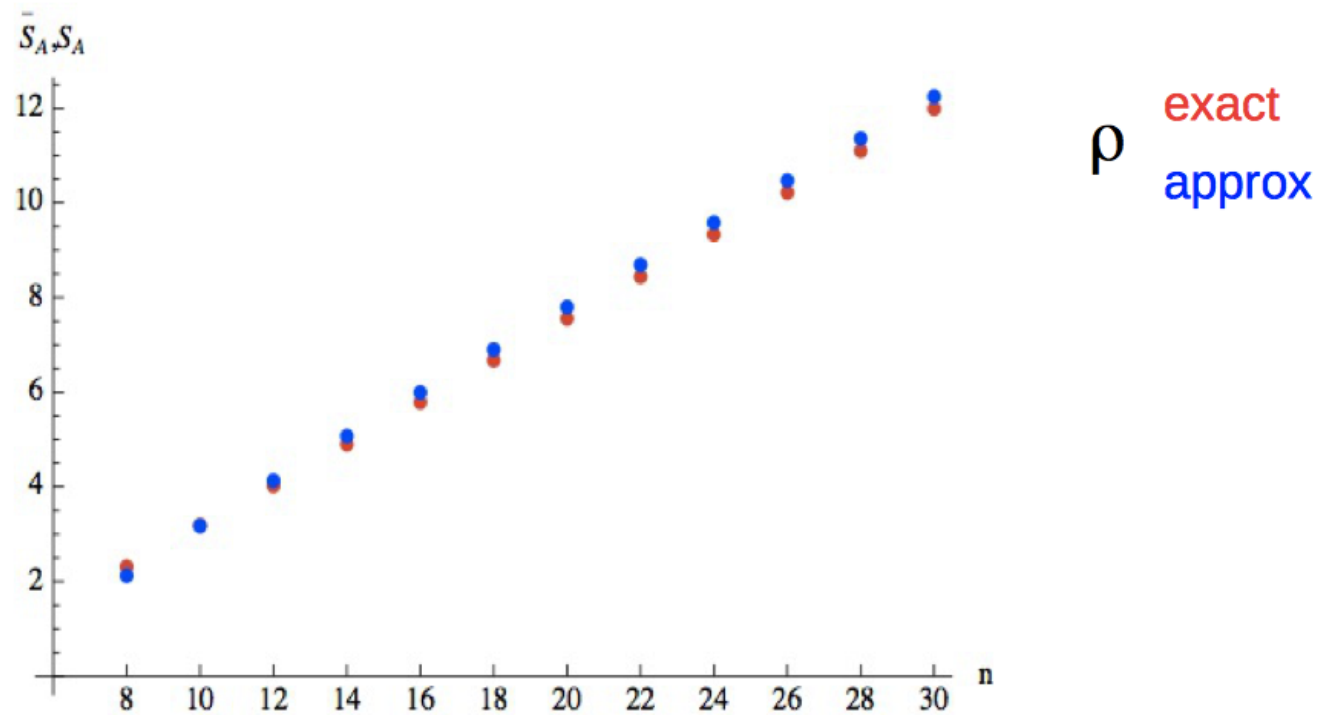
# Density matrix of the Prime state



$$\rho_n = -Tr_B |P(n)\rangle\langle P(n)|$$

$$S_n = -Tr_A (\rho_n \log \rho_n)$$

# Entanglement entropy of the Prime state



Volumen law entropy

$$S_A \approx c_\pi \frac{n}{2} + cte$$

$$c_\pi \approx 0.886(1)$$

## Scaling of entanglement entropy

$$S \sim n - \text{const}$$

Random states

$$S \sim .8858 n + \text{const}$$

Prime state

$$S \sim n^{\frac{d-1}{d}} + \text{const}$$

Area law in d-dimensions

$$S \sim \frac{c}{3} \log n + \text{const}$$

Critical scaling in d=1  
at quantum phase transitions

$$S \sim \log(\xi) = \text{const}$$

Finitely correlated states  
away from criticality

Entropies for Random Ensembles Lubkin (78), Don Page (93)

$$|\psi\rangle = \sum_C \psi(C) |C\rangle$$

$\psi(C)$  (real or complex) uniformly distributed in the Hilbert space

$$\langle S_{vN} \rangle = \langle S_n \rangle = \ln(|\mathcal{H}_A|) \sim V_A$$

Entropies for Random Positive Ensembles Grover, Fisher (14)

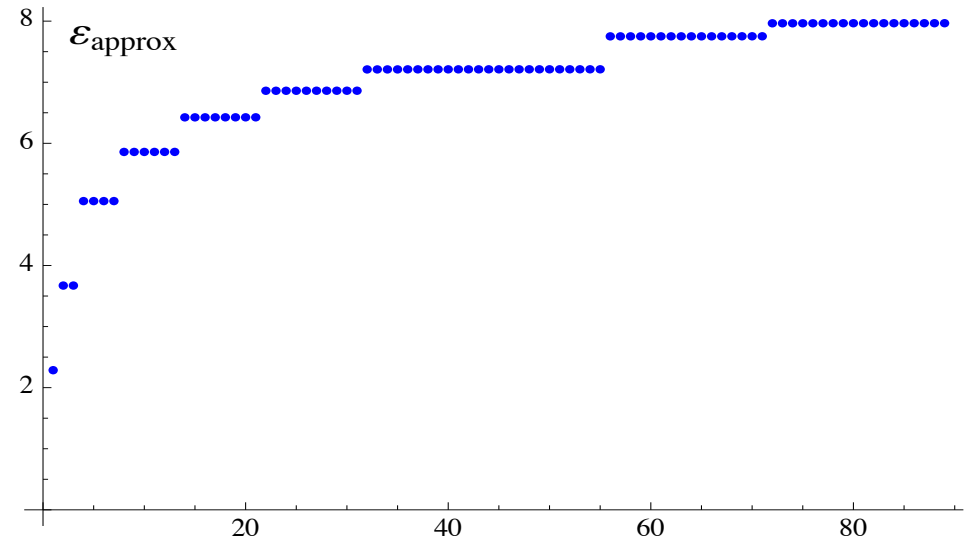
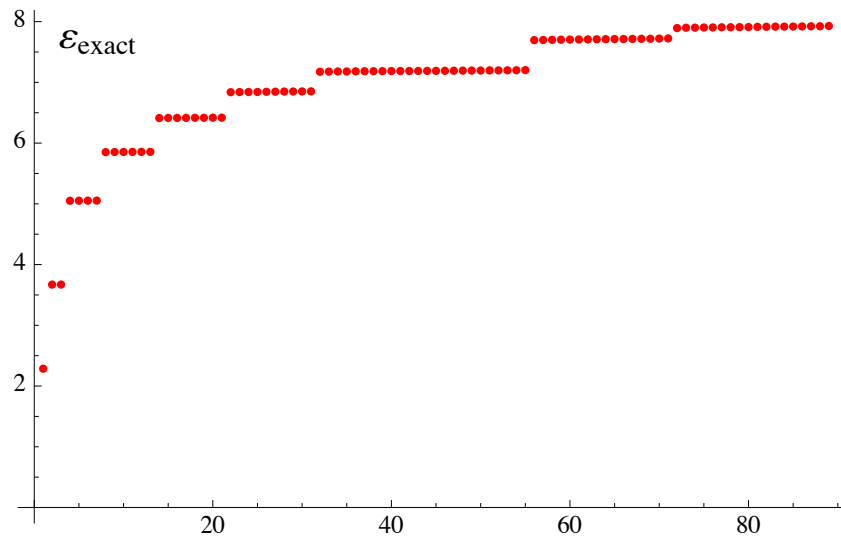
$$||\psi| \rangle = \sum_C |\psi(C)| |C\rangle$$

$$S_n(\langle \rho_A \rangle) = \begin{cases} \frac{n}{n-1} \ln\left(\frac{\pi}{2}\right) & \text{if } n > 1, \\ \left(1 - \frac{2}{\pi}\right) \ell_A^d & \text{if } n = 1, \\ \ell_A^d & \text{if } n \leq 1, \end{cases}$$

# Entanglement Spectrum

$\lambda_i$  eigenvalues of  $\rho_A$

$$\varepsilon_i = -\log \lambda_i$$



## The prime state and Hardy-Littlewood conjecture

Approximate reduced density matrix

$$\bar{\rho}_A = \frac{1}{d} (I + \ell_m C_m)$$

$C_m$  Toeplitz matrix using the Hardy-Littlewood constants

$$C_m(i, j) = C(2(i - j)), \quad i, j = 1, \dots, d = 2^{m-1}$$

Quantum  
Entanglement



Pairwise  
Correlations

The entanglement spectrum follows from the spectrum of  $C_m$

square free number  $n = p_1 p_2 \cdots p_r, \quad p_i \neq p_j$

Moebius function  $\mu(n) = (-1)^r \text{ or } 0 \text{ else}$

Positive Eigenvalues:  $\gamma_j = 2^m \left( \frac{\mu(j)}{\phi(j)} \right)^2, \quad j = 1, 3, 5, \dots$

Degeneracy:  $\phi(j)$  Euler totient function

$n$	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	33	35	37	39
$\phi(n)$	1	2	4	6	6	10	12	8	16	18	12	22	20	18	28	30	20	24	36	24
$\hat{\gamma}_n$	1	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{36}$	0	$\frac{1}{100}$	$\frac{1}{144}$	$\frac{1}{64}$	$\frac{1}{256}$	$\frac{1}{324}$	$\frac{1}{144}$	$\frac{1}{484}$	0	0	$\frac{1}{784}$	$\frac{1}{900}$	$\frac{1}{400}$	$\frac{1}{576}$	$\frac{1}{1296}$	$\frac{1}{576}$

**Conjecture:**  $c_\pi = H_2\left(\frac{3}{\pi^2}\right)$

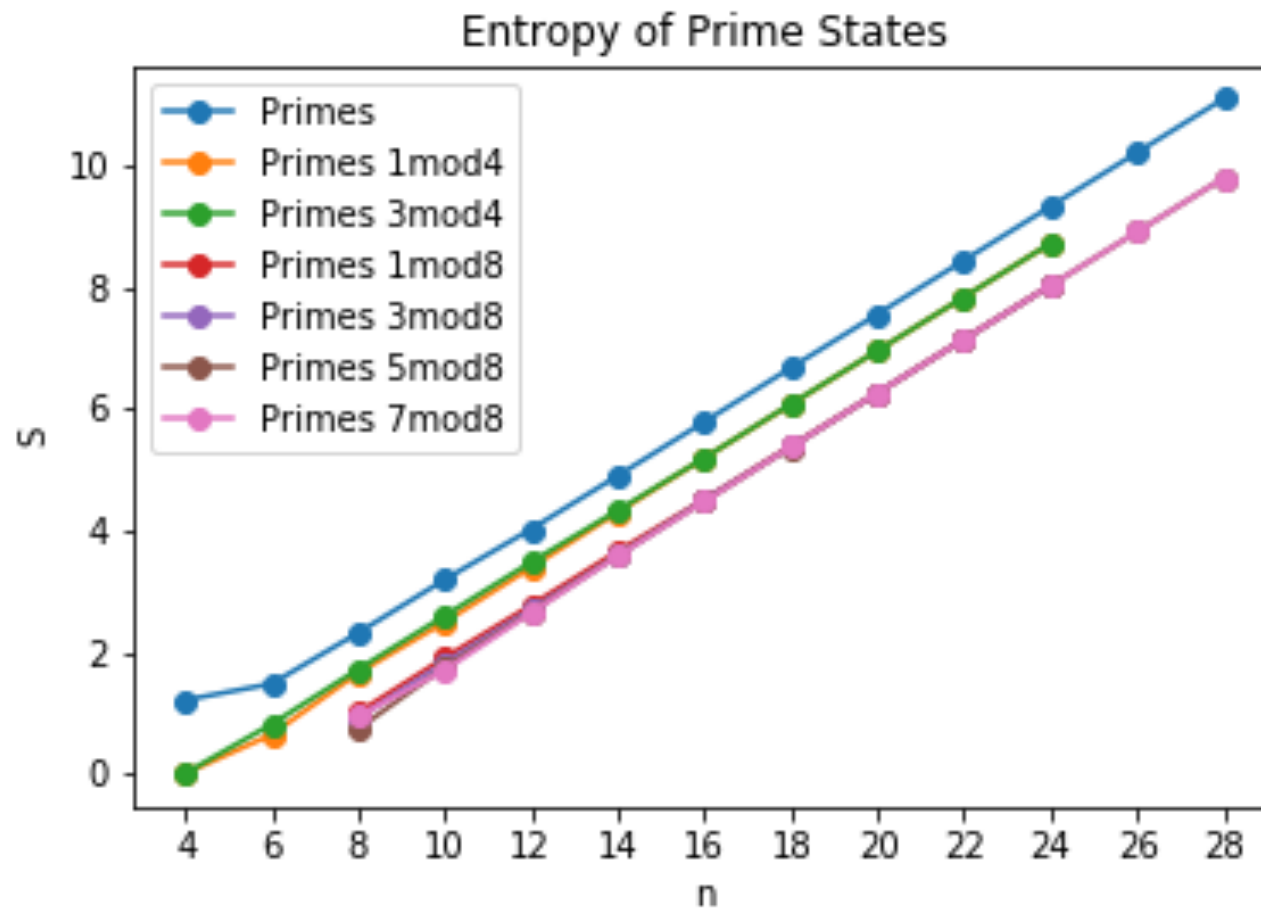
$$H_2(x) = -x \log_2 x - (1-x) \log_2(1-x)$$

$$\frac{3}{\pi^2} = \frac{1}{2 \zeta(2)} : \text{density of odd square free integers}$$

$$c_\pi^{num} = 0.886(1), \quad c_\pi^{th} = 0.886082 \dots$$



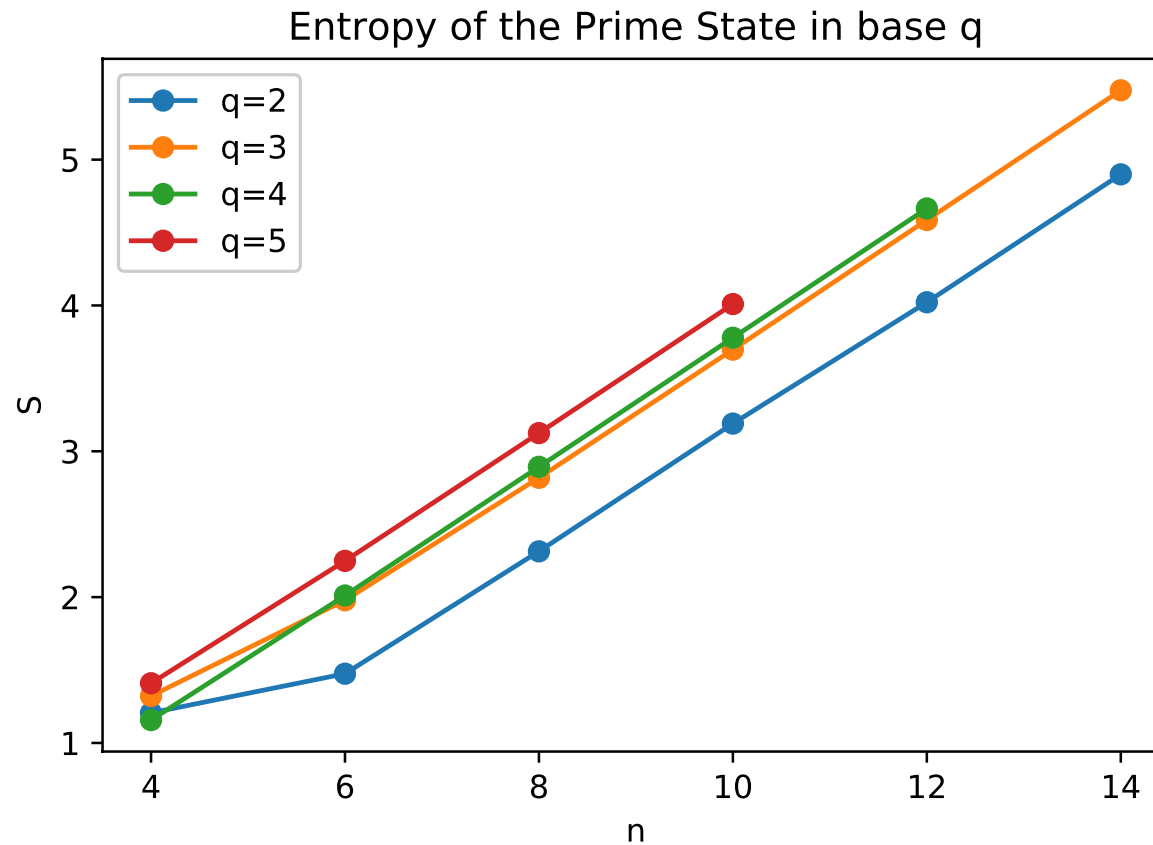
# Entropies of primes in arithmetics progressions



Prime 1 mod 4  
 Prime 3 mod 4

$$S_A \approx c_\pi \frac{n}{2} + cte$$

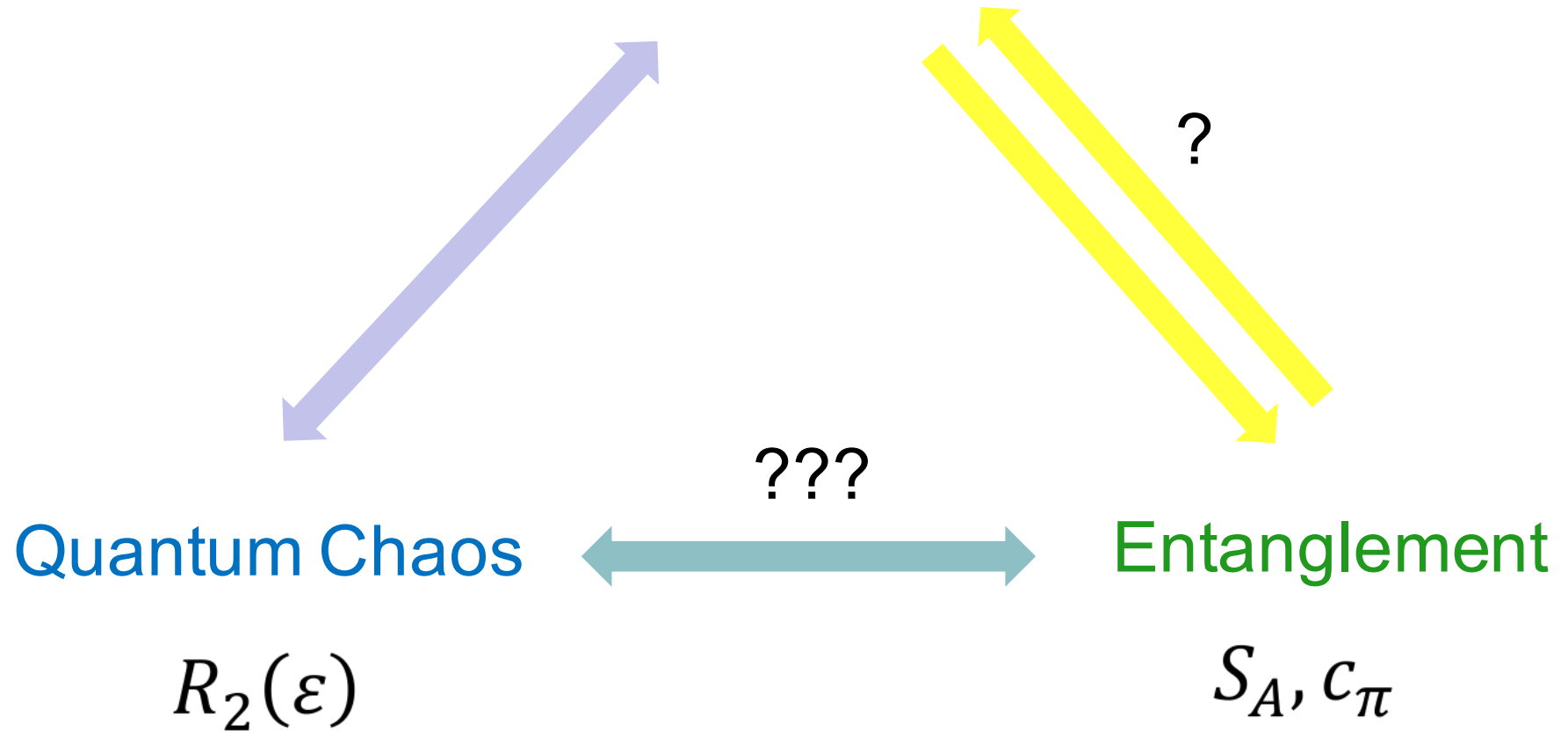
# Entropies of primes using qudits



$C_\pi$  : entanglement prime constant

**CONCLUSIONS**

# Hardy-Littlewood conjecture



## References

”Quantum Computation of prime number functions”,  
J.I. Latorre and G.S., Quan. Info. Comm. 2014.

”There is entanglement in the primes”,  
J.I. Latorre and G.S., Quan. Info. Comm. 2015.

”Five experimental Tests on the 5-Qubit IBM Quantum  
Computer”,  
D. García-Martín and G.S., J. Applied Maths and Phys. 2018

Congratulations

Congratulations

The best is yet to come